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An Incomplete Lipschitz-Hankel Integral of K_0

Part I

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Engineering Services Division

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An incomplete Lipschitz-Hankel Integral of K_0 and related integrals are given in terms of elementary, cylindrical, and Kampé de Fériet functions. Some of the properties of these Kampé de Fériet functions are derived

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AN INCOMPLETE LIPSCHITZ-HANKEL INTEGRAL OF K_0
PART I

INTRODUCTION

An incomplete Lipschitz-Hankel integral of cylindrical functions of order zero, C_0, may be defined by

$$C_{e_0}(a, z) \equiv \int_0^z e^{at} C_0(t)\, dt$$

Of interest in applications are the functions $J_{e_0}(a, z)$, $I_{e_0}(a, z)$, and $N_{e_0}(a, z)$ where J denotes the Bessel function of the first kind, I denotes the modified Bessel function, and N denotes the Bessel function of the second kind or Neumann function. $J_{e_0}(a, z)$ and $N_{e_0}(a, z)$ occur in problems in the theory of diffraction in optical apparatus [1, p. 227]. The function $I_{e_0}(a, z)$ plays an important role in the study of oscillating wings in supersonic flow and arises in the study of resonant absorption in media with finite dimensions [1, p. 195].

In this report we are interested in

$$K_{e_0}(a, z) \equiv \int_0^z e^{at} K_0(t)\, dt \tag{1}$$

where K denotes the MacDonald function or Bessel function of imaginary argument. We shall show that $K_{e_0}(a, z)$ can be written in closed form in terms of elementary functions, K_0, K_1, and Kampé de Fériet double hypergeometric functions. As an application it shall be shown that $K_{e_0}(a, z)$ occurs when the statistical distribution of the maxima of a random function is applied to the amplitude of a sine wave in order to calculate the distribution of its ordinate. This latter distribution is of interest in the study of the scattered coherent reflected field from the sea surface [2].

Moreover we derive formulas for several integrals that are not readily available, and we exhibit some of the properties of the Kampé de Fériet functions associated with $K_{e_0}(a, z)$.

PRELIMINARY DEFINITIONS

The Pochhammer symbol $(a)_n$ is defined for nonnegative integers n as a ratio of gamma functions:

$$\begin{aligned} (a)_n &\equiv \Gamma(a + n)/\Gamma(a) = a(a + 1) \ldots (a + n - 1) \\ (a)_0 &\equiv 1 \end{aligned} \tag{2}$$

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Following Srivastava and Panda [3, p. 63] we define the Kampé de Fériet double hypergeometric functions:

$$F^{p:q;k}_{l:m;n}\left[\begin{matrix}(a_p):(b_q);\ (c_k);\\ (\alpha_l):(\beta_m);\ (\gamma_n);\end{matrix}\ x,y\right] \equiv \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}$$

where the Pochhamner symbols $(a)_n$ are defined by Eq. (2). For convergence

$$p+q<l+m+1,\ p+k<l+n+1,\ |x|<\infty,\ |y|<\infty, \text{ or}$$

$$p+q=l+m+1,\ p+k=l+n+1, \text{ and}$$

$$\begin{cases}|x|^{1/(p-l)}+|y|^{1/(p-l)}<1 & p>l\\ \max\{|x|,|y|\}<1 & p\leqslant l\end{cases}$$

As special cases we define

$$L[\alpha,\beta;\gamma,\delta;x,y] \equiv \sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\beta)_n}{(\gamma)_{m+n}(\delta)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \qquad |x|<\infty,\ |y|<\infty \tag{3}$$

$$M[\alpha,\beta;\gamma,\delta;x,y] \equiv \sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_n}{(\gamma)_{m+n}(\delta)_{m}}\frac{x^m}{m!}\frac{y^n}{n!} \qquad |x|<\infty,\ |y|<1 \tag{4}$$

We may then write

$$L[\alpha,\beta;\gamma,\delta;x,y] = F^{0:1;1}_{2:0;0}\left[\begin{matrix}-\ :\alpha;\beta;\\ \gamma,\delta:-;-;\end{matrix}\ x,y\right]$$

$$M[\alpha,\beta;\gamma,\delta;x,y] = F^{1:0;1}_{1:1;0}\left[\begin{matrix}\alpha:-;\beta;\\ \gamma:\delta;-;\end{matrix}\ x,y\right]$$

SOME ELEMENTARY PROPERTIES OF $M[\alpha,\beta;\gamma,\delta;x,y]$

Substituting [4, p. 266]

$$\frac{(\alpha)_p}{(\gamma)_p} = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{p+\alpha-1}(1-t)^{\gamma-\alpha-1}dt$$

where Re $\gamma >$ Re $\alpha > 0$, and $p = m + n$ into Eq. (4), we deduce an integral representation for M:

$$M[\alpha, \beta; \gamma, \delta; x, y] = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 {}_0F_1[-; \delta; xt] t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-yt)^{-\beta} dt$$

$$= \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} x^{-\frac{\delta-1}{2}} \int_0^1 I_{\delta-1}(2\sqrt{xt})\, t^{\alpha-\frac{\delta+1}{2}} (1-t)^{\gamma-\alpha-1}(1-yt)^{-\beta} dt$$

Here we have used the equation

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} {}_0F_1[-; \nu+1; z^2/4] \tag{5}$$

We obtain directly from Eq. (4) the generating relation

$$M[\alpha, \beta; \gamma, \delta; x, y] = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n(\delta)_n} \frac{x^n}{n!} {}_2F_1[n+\alpha, \beta; n+\gamma; y] \tag{6}$$

We now prove the following

THEOREM: Suppose $-1 < \mathrm{Re}(\gamma - \alpha - \beta) < 0$, $|\arg y| < \pi$, $|\arg(1-y)| < \pi$. Then for $y \to 1$,

$$M[\alpha, \beta; \gamma, \delta; x, y] = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} {}_1F_2[\alpha; \gamma-\beta, \delta; x]$$

$$+ \frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)} (1-y)^{\gamma-\alpha-\beta} {}_0F_1[-; \delta; x] + O(1-y) \tag{7}$$

or

$$M[\alpha, \beta; \gamma, \delta; x, y] = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} {}_1F_2[\alpha; \gamma-\beta, \delta; x] + O(1-y)$$

$$+ \Gamma(\alpha+\beta-\gamma) \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\alpha)\Gamma(\beta)} x^{-\frac{\delta-1}{2}} (1-y)^{\gamma-\alpha-\beta} I_{\delta-1}(2\sqrt{x}) \tag{8}$$

Proof: The following result is found in [4, Eq. (9.5.7), p. 249]: for $\alpha + \beta - \gamma \neq 0, \pm 1, \pm 2, \ldots$, $|\arg z| < \pi$, $|\arg(1-z)| < \pi$

$${}_2F_1[\alpha, \beta; \gamma; z] = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} {}_2F_1[\alpha, \beta; 1+\alpha+\beta-\gamma; 1-z]$$

$$+ (1-z)^{\gamma-\alpha-\beta} \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} {}_2F_1[\gamma-\alpha, \gamma-\beta; 1-\alpha-\beta+\gamma; 1-z]$$

Hence

$$
{}_2F_1[n+\alpha,\beta;n+\gamma;y] = \frac{\Gamma(n+\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(n+\gamma-\beta)}\,{}_2F_1[n+\alpha,\beta;1+\alpha+\beta-\gamma;1-y]
$$

$$
+(1-y)^{\gamma-\alpha-\beta}\frac{\Gamma(n+\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(n+\alpha)\Gamma(\beta)}\,{}_2F_1[\gamma-\alpha,n+\gamma-\beta;1-\alpha-\beta+\gamma;1-y]
$$

Now suppose that $-1 < \mathrm{Re}(\gamma-\alpha-\beta) < 0$. Then for $y \to 1$ we have

$$
{}_2F_1[n+\alpha,\beta;n+\gamma;y] = \frac{\Gamma(n+\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(n+\gamma-\beta)} + \frac{\Gamma(n+\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(n+\alpha)\Gamma(\beta)}(1-y)^{\gamma-\alpha-\beta} + O(1-y)
$$

$$
= \frac{(\gamma)_n}{(\gamma-\beta)_n}\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\beta)\Gamma(\gamma-\alpha)} + \frac{(\gamma)_n}{(\alpha)_n}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-y)^{\gamma-\alpha-\beta} + O(1-y)
$$

Substituting this result into Eq. (6) gives

$$
M[\alpha,\beta;\gamma,\delta;x,y] = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma-\beta)_n(\delta)_n}\frac{x^n}{n!}
$$

$$
+\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-y)^{\gamma-\alpha-\beta}\sum_{n=0}^{\infty}\frac{x^n}{(\delta)_n n!} + O(1-y)
$$

from which we obtain Eq. (7). Then using Eq. (5) we obtain Eq. (8).

Employing series rearrangement we deduce

$$
M[\alpha,\beta;\gamma,\delta;x,tx] = \sum_{p=0}^{\infty}\frac{(\alpha)_p(\beta)_p}{(\gamma)_p}\frac{x^p}{p!}\,t^p\,{}_1F_2[-p;\delta,1-\beta-p;1/t] \tag{9}
$$

Using a general result of Srivastava [3, Eq. (30), p. 145] we find Eq. (9) in a different form, *viz*,

$$
M[\alpha,\beta;\gamma,\delta;tx,t] = \sum_{p=0}^{\infty}\frac{(\alpha)_p(\beta)_p}{(\gamma)_p}\frac{t^p}{p!}\,{}_1F_2[-p;\delta,1-\beta-p;x]
$$

From Eq. (9) it follows that

$$
M[\alpha,\beta;\gamma,\delta;x,tx] = \sum_{p=0}^{\infty}\frac{(\alpha)_p}{(\gamma)_p(\delta)_p}\frac{x^p}{p!}\,{}_3F_0[\beta,-p,1-\delta-p;-;t] \tag{10}
$$

Equation (10) may be obtained directly from [3, Eq. (60.ii), p. 194].

We remark that it may be shown that $M[\alpha, 1; \gamma, \delta; x, y]$ converges on the unit circle $|y| = 1$ if and only if ${}_2F_1[\alpha, 1; \gamma; y]$ converges on $|y| = 1$.

SOME ELEMENTARY PROPERTIES OF $L[\alpha, \beta; \gamma, \delta; x, y]$

Using series rearrangement we find

$$L[\alpha, \beta; \gamma, \delta; x, tx] = \sum_{p=0}^{\infty} \frac{(\alpha)_p}{(\gamma)_p (\delta)_p} \frac{x^p}{p!} \, {}_2F_1[\beta, -p; 1 - \alpha - p; t]$$

This can also be obtained from [3, Eq. (30), p. 145] in a different form. Using Vandermonde's theorem [5, Eq. (1.7.7), p. 28]

$${}_2F_1[a, -p; c; 1] = (c - a)_p/(c)_p$$

$${}_2F_1[\beta, -p; 1 - \alpha - p; 1] = \frac{(1 - \alpha - \beta - p)_p}{(1 - \alpha - p)_p} = \frac{(\alpha + \beta)_p}{(\alpha)_p}$$

so that we have a reduction formula for L, *viz*,

$$L[\alpha, \beta; \gamma, \delta; x, x] = \sum_{p=0}^{\infty} \frac{(\alpha + \beta)_p}{(\gamma)_p (\delta)_p} \frac{x^p}{p!} = {}_1F_2[\alpha + \beta; \gamma, \delta; x] \tag{11}$$

This result can be obtained also by using the following general result of Srivastava [3, Eq. (20), p. 55] applied to Eq. (3):

$$\sum_{m,n=0}^{\infty} c_{m+n} (\rho)_m (\sigma)_n \frac{x^{m+n}}{m! n!} = \sum_{n=0}^{\infty} c_n (\rho + \sigma)_n \frac{x^n}{n!}$$

provided each series is absolutely convergent.

We obtain directly from Eq. (3) the generating relation

$$L[\alpha, \beta; \gamma, \delta; x, y] = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_m (\delta)_m} \frac{x^m}{m!} \, {}_1F_2[\beta; m + \gamma, m + \delta; y] \tag{12}$$

Finally, using [3, Eq. (43), p. 150] we obtain

$$L[\alpha, \beta; \gamma, \delta; -x, x \tan^2\theta] = (\cos^2\theta)^{\beta} \sum_{n=0}^{\infty} \frac{(\beta)_n (\sin^2\theta)^n}{n!} \, {}_1F_2[\alpha - n; \gamma, \delta; -x]$$

A CLOSED FORM FOR $K_{e_0}(a, z)$

From Eq. (1) we write

$$K_{e_0}(\alpha/\beta, \beta) = \beta \int_0^1 e^{\alpha t} K_0(\beta t)\, dt \tag{13}$$

Using [6, p. 89] we find the following formulas:

$$\int_0^1 s^m K_0(zs)\, ds = \frac{K_0(z)}{m+1}\, {}_1F_2\left[1;\ \frac{m+1}{2},\ \frac{m+3}{2};\ \frac{z^2}{4}\right]$$

$$+ \frac{z K_1(z)}{(m+1)^2}\, {}_1F_2\left[1;\ \frac{m+3}{2},\ \frac{m+3}{2};\ \frac{z^2}{4}\right] \qquad m = 0,2,4,\ldots \tag{14}$$

$$\int_0^1 s^m K_0(zs)\, ds = \frac{2^{m-1}\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m+1}{2}\right)}{z^{m+1}}$$

$$- \frac{(m-1)K_0(z)}{z^2}\, {}_3F_0\left[1,\ \frac{1-m}{2},\ \frac{3-m}{2};\ -;\ \frac{4}{z^2}\right]$$

$$- \frac{K_1(z)}{z}\, {}_3F_0\left[1,\ \frac{1-m}{2},\ \frac{1-m}{2};\ -;\ \frac{4}{z^2}\right] \qquad m = 1,3,5,\ldots \tag{15}$$

Integrating term by term we find

$$\int_0^1 \exp(\alpha t) K_0(\beta t)\, dt = \int_0^1 \sum_{n=0}^{\infty} \frac{\alpha^n t^n}{n!} K_0(\beta t)\, dt = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \int_0^1 t^n K_0(\beta t)\, dt$$

$$= \sum_{n=0,2,4,} \frac{\alpha^n}{n!} \int_0^1 t^n K_0(\beta t)\, dt + \sum_{n=1,3,5,} \frac{\alpha^n}{n!} \int_0^1 t^n K_0(\beta t)\, dt$$

$$= \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!} \int_0^1 t^{2n} K_0(\beta t)\, dt + \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!} \int_0^1 t^{2n+1} K_0(\beta t)\, dt$$

so that using Eqs. (14) and (15)

$$\int_0^1 \exp(\alpha t)\, K_0(\beta t)\,dt = \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!}\, \frac{K_0(\beta)}{2n+1}\; {}_1F_2\left[1;\, \frac{2n+1}{2},\, \frac{2n+3}{2};\, \frac{\beta^2}{4}\right]$$

$$+ \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!}\, \frac{\beta K_1(\beta)}{(2n+1)^2}\; {}_1F_2\left[1;\, \frac{2n+3}{2},\, \frac{2n+3}{2};\, \frac{\beta^2}{4}\right]$$

$$+ \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!}\, 2^{2n}\, \frac{\Gamma(n+1)\Gamma(n+1)}{\beta^{2n+2}}$$

$$- \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!}\, (2n)\, \frac{K_0(\beta)}{\beta^2}\; {}_3F_0[1,\, -n,\, 1-n;\, -;\, 4/\beta^2]$$

$$- \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!}\, \frac{K_1(\beta)}{\beta}\; {}_3F_0[1,\, -n,\, -n;\, -;\, 4/\beta^2] \qquad (16)$$

We shall consider each of the above five sums in the order in which they appear. We find

$$\sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n+1)!}\; {}_1F_2[1;\, n+1/2,\, n+3/2;\, \beta^2/4]$$

$$= \sum_{n=0}^{\infty} \frac{1}{(3/2)_n}\, \frac{(\alpha^2/4)^n}{n!}\; {}_1F_2[1,\, n+1/2,\, n+3/2;\, \beta^2/4] = L\,[1/2,\, 1;\, 1/2,\, 3/2;\, \alpha^2/4,\, \beta^2/4];$$

$$\sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n+1)(2n+1)!}\; {}_1F_2[1;\, n+3/2,\, n+3/2;\, \beta^2/4]$$

$$= \sum_{n=0}^{\infty} \frac{(1/2)_n}{(3/2)_n (3/2)_n}\, \frac{(\alpha^2/4)^n}{n!}\; {}_1F_2[1,\, n+3/2,\, n+3/2;\, \beta^2/4] = L\,[1/2,\, 1;\, 3/2,\, 3/2,\, \alpha^2/4,\, \beta^2/4]$$

where in the latter two cases we have used Eq. (12);

$$\sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!}\, 2^{2n}\, \frac{\Gamma(n+1)\Gamma(n+1)}{\beta^{2n+2}} = \frac{\sin^{-1}(\alpha/\beta)}{\sqrt{\beta^2-\alpha^2}} \qquad |\alpha/\beta| \leqslant 1, \quad \alpha \neq \pm\beta$$

wh[illegible] have used [9, Eq. (9.121-14), p. 1041] the result ${}_2F_1[1, 1; 3/2; \sin^2 z] = z/\sin z \cos z$;

$$\sum_{n=0} \frac{\alpha^{2n+1}}{(2n \cdots} \cdots n)\, {}_3F_0[1, -n, 1-n; -; 4/\beta^2] = \sum_{n=0}^{\infty} \frac{\alpha^{2n+3}}{(2n+3)!} (2n+2)\, {}_3F_0[1, -1-n, -n; -; 4/\beta^2]$$

$$= \sum_{n=0} \frac{1}{(5/2)_n} \frac{(\alpha^2/4)^n}{n!}\, {}_3F_0[1, -1-n, -n; -; 4/\beta^2] = \frac{\alpha^3}{3} M \left[2, 1; \frac{5}{2}, 2; \frac{\alpha^2}{4}, \frac{\alpha^2}{\beta^2}\right];$$

and finally

$$\sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!}\, {}_3F_0\, [1, -n, -n; -; 4/\beta^2]$$

$$= \alpha \sum_{n=0}^{\infty} \frac{1}{(3/2)_n} \frac{(\alpha^2/4)^n}{n!}\, {}_3F_0[1, -n, -n; -; 4/\beta^2] = \alpha\, M \left[1, 1; \cdots \frac{\alpha^2}{4}, \frac{\alpha^2}{\beta^2}\right]$$

where in the latter two cases we have used Eq. (10).

Defining

$$L_0(x, y) \equiv \sum_{m,n=0}^{\infty} \frac{(1/2)_m (1)_n}{(3/2)_{m+n} (3/2)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} = L[1/2, 1; 3/2, 3/2; x, y]$$

$$L_1(x, y) \equiv \sum_{m,n=0}^{\infty} \frac{(1/2)_m (1)_n}{(1/2)_{m+n} (3/2)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} = L\, [1/2, 1; 1/2, 3/2; x, y]$$

$$M_0(x, y) \equiv \sum_{m,n=0}^{\infty} \frac{(1)_{m+n}}{(3/2)_{m+n}} \frac{(1)_n}{(1)_m} \frac{x^m}{m!} \frac{y^n}{n!} = M\, [1, 1; 3/2, 1; x, y]$$

$$M_1(x, y) \equiv \sum_{m,n=0}^{\infty} \frac{(2)_{m+n}}{(5/2)_{m+n}} \frac{(1)_n}{(2)_m} \frac{x^m}{m!} \frac{y^n}{n!} = M[2, 1; 5/2, 2; x, y]$$

we have from Eq. (16) and the above results

$$\int_0^1 \exp(\alpha t) K_0(\beta t)\, dt = K_1(\beta) \left[\beta L_0(\alpha^2/4, \beta^2/4) - \frac{\alpha}{\beta} M_0(\alpha^2/4, \alpha^2/\beta^2)\right]$$

$$+ K_0(\beta) \left[L_1(\alpha^2/4, \beta^2/4) - \frac{\alpha^3}{3\beta^2} M_1(\alpha^2/4, \alpha^2/\beta^2)\right] + \frac{\sin^{-1}(\alpha/\beta)}{\sqrt{\beta^2 - \alpha^2}} \qquad (17)$$

which we may write using Eq. (13)

$$K_{e_0}(a, z) = z\,K_1(z)\left[z\,L_0\left[\frac{a^2z^2}{4}, \frac{z^2}{4}\right] - a\,M_0\left[\frac{a^2z^2}{4}, a^2\right]\right]$$

$$+ z\,K_0(z)\left[L_1\left[\frac{a^2z^2}{4}, \frac{z^2}{4}\right] - \frac{a^3z}{3}M_1\left[\frac{a^2z^2}{4}, a^2\right]\right] + \frac{\sin^{-1} a}{\sqrt{\;- a^2}} \tag{18}$$

We have then given $K_{e_0}(a, z)$ in terms of elementary, MacDonald, and Kampé de Fériet functions.

We remark that in view of Eq. (6) and the definitions of M_0 and M_1

$$M_0(x, y) = \sum_{n=0}^{\infty} \frac{1}{(3/2)_n}\frac{x^n}{n!}\,{}_2F_1[1, n+1; n+3/2; y]$$

$$M_1(x, y) = \sum_{n=0}^{\infty} \frac{1}{(5/2)_n}\frac{x^n}{n!}\,{}_2F_1[1, n+2; n+5/2; y]$$

Since each of the Gauss hypergeometric functions above is conditionally convergent on the unit circle $|y| = 1$ except at $y = 1$ we see that $M_0(x, y)$ and $M_1(x, y)$ are conditionally convergent on $|y| = 1$ except at $y = 1$.* Hence Eq. (17) is valid for $|\alpha/\beta| \leqslant 1$, $\alpha \neq \pm\beta$ and Eq. (18) is valid for $|a| \leqslant 1$, $a \neq \pm 1$. We shall show shortly that Eq. (17) is valid in the limit even when $\alpha = \pm\beta$. See Ref. 1 for other representations of $K_{e_0}(a, z)$.

In a future report (Part II) it shall be shown that Eq. (18) is easily extended to the entire complex a-plane in terms of elementary, MacDonald and Kampé de Fériet functions.

KING'S INTEGRAL

Using properties of L and M we have derived earlier we shall derive (a formula for) King's integral [6, Eq. (12), p. 123]:

$$\int_0^{\alpha} \exp t\, K_0(t)\,dt = \alpha \exp \alpha\,[K_0(\alpha) + K_1(\alpha)] - 1 \tag{19}$$

that is we shall show Eq. (17) is valid in the limit for $\alpha = \beta$. Using Eq. (8) we find for $\alpha \to \beta$

$$M_0(\alpha^2/4, \alpha^2/\beta^2) = \frac{\pi}{2}\frac{I_0(\alpha)}{\sqrt{1 - \alpha^2/\beta^2}} - \cosh \alpha + O(1 - \alpha^2/\beta^2)$$

$$M_1(\alpha^2/4, \alpha^2/\beta^2) = \frac{3}{\alpha}\left[\frac{\pi}{2}\frac{I_1(\alpha)}{\sqrt{1 - \alpha^2/\beta^2}} - \sinh \alpha + O(1 - \alpha^2/\beta^2)\right]$$

*Also see remark on top of p. 5

Substituting these equations into Eq. (17) gives

$$\int_0^1 \exp(\alpha t) K_0(\beta t)\, dt = \frac{\sin^{-1}(\alpha/\beta) - \frac{\pi}{2}[\alpha K_1(\beta)\, I_0(\alpha) + (\alpha^2/\beta)\, K_0(\beta)\, I_1(\alpha)]}{\sqrt{\beta^2 - \alpha^2}} + \frac{\alpha}{\beta} K_1(\beta) \cosh\alpha$$

$$+ \frac{\alpha^2}{\beta^2} K_0(\beta) \sinh\alpha + K_0(\beta) L_1(\alpha^2/4, \beta^2/4) + \beta K_1(\beta)\, L_0(\alpha^2/4, \beta^2/4) + O(1 - \alpha^2/\beta^2) \qquad (20)$$

Using the reduction formula Eq. (11) for L we deduce

$$L_0(x^2/4, x^2/4) = \frac{\sinh x}{x}$$

$$L_1(x^2/4, x^2/4) = \cosh x$$

Now holding β fixed and letting $\alpha \to \beta$ we obtain after simplification

$$\int_0^1 \exp(\beta t) K_0(\beta t)\, dt = [K_0(\beta) + K_1(\beta)] \exp\beta + \lim_{\alpha \to \beta} J(\alpha, \beta)$$

where $J(\alpha, \beta)$ is the first term on the right-hand side of Eq. (20). We find however that

$$\lim_{\alpha \to \beta} [\text{numerator } J(\alpha, \beta)] = \frac{\pi}{2}[1 - \beta K_1(\beta)\, I_0(\beta) - \beta K_0(\beta)\, I_1(\beta)] = 0$$

$$\lim_{\alpha \to \beta} [\text{denominator } J(\alpha, \beta)] = 0$$

so that on applying L'hospital's rule we have

$$\lim_{\alpha \to \beta} J(\alpha, \beta) = -1/\beta$$

Hence

$$\int_0^1 \exp(\beta t) K_0(\beta t)\, dt = [K_0(\beta) + K_1(\beta)] \exp\beta - 1/\beta$$

and a simple transformation now gives Eq. (19). We may perform a similar analysis for $\alpha \to -\beta$ to obtain

$$\int_0^1 \exp(-\beta t) K_0(\beta t)\, dt = [K_0(\beta) - K_1(\beta)] \exp(-\beta) + 1/\beta$$

A DISTRIBUTION FOR THE ELEVATION OF A SINE WAVE

Consider the random variable $y = H \sin\theta$, where H is a random variable with density $K(H, \epsilon)$, $|H| < \infty$, and θ is a random variable, independent of H, with density

$$U(\theta) = \pi^{-1} \qquad |\theta| \leqslant \pi/2$$
$$= 0 \qquad |\theta| > \pi/2$$

Let $D(y, \epsilon)$ be the density function for y. It is shown in Ref. 2 that

$$D(y, \epsilon) = \frac{1}{\pi}\int_{-\infty}^{-|y|} \frac{K(H, \epsilon)dH}{\sqrt{H^2 - y^2}} + \frac{1}{\pi}\int_{|y|}^{\infty} \frac{K(H, \epsilon)\, dH}{\sqrt{H^2 - y^2}} \tag{21}$$

Rice [7] and Cartwright and Longuet-Higgins [8] have derived an expression for the statistical distribution of the maxima of a random function that may be expressed in the form

$$K(H, \epsilon) = \frac{\epsilon}{\sigma_H\sqrt{2\pi}} \exp\left(\frac{-H^2}{2\epsilon^2\sigma_H^2}\right) + \frac{\sqrt{1-\epsilon^2}}{2\sigma_H^2} H \exp\left(\frac{-H^2}{2\sigma_H^2}\right)\left[1 + \operatorname{erf}\left(\frac{\sqrt{2}}{2}\frac{H}{\sigma_H}\frac{\sqrt{1-\epsilon^2}}{\epsilon}\right)\right] \tag{22}$$

Here σ_H is the standard deviation of H, and $0 < \epsilon < 1$ is known as the spectral width parameter. It is shown in Ref. 2 that the standard deviation σ of y is given by

$$\sigma = \sigma_H/(\sqrt{2}\eta)$$

where η is defined by

$$\eta \equiv \left[1 + \frac{\pi}{2}(1 - \epsilon^2)\right]^{-1/2}$$

Substituting Eq. (22) into Eq. (21) and using the latter result gives

$$D(y, \epsilon) = \frac{\epsilon}{2\pi^{3/2}\eta\sigma} \exp\left(\frac{-y^2}{8\epsilon^2\eta^2\sigma^2}\right) K_0\left(\frac{y^2}{8\epsilon^2\eta^2\sigma^2}\right) + \frac{\sqrt{1-\epsilon^2}}{\pi\eta\sigma} \exp\left(\frac{-y^2}{4\eta^2\sigma^2}\right) \Psi\left(\frac{\sqrt{1-\epsilon^2}}{\epsilon}, \frac{y}{2\eta\sigma}\right) \tag{23}$$

where the function $\Psi(k, u)$ is defined by

$$\Psi(k, u) \equiv \int_0^\infty \exp(-s^2) \operatorname{erf}(k\sqrt{u^2 + s^2})\, ds \tag{24}$$

For real u and k it is shown in Ref. 2 that

$$\pi^{1/2}\int_0^\infty \exp(-s^2) \operatorname{erf}(k\sqrt{u^2+s^2})\, ds = \tan^{-1}k + \frac{k}{1+k^2}\int_0^{\frac{1}{2}u^2(1+k^2)} \exp\left(\frac{1-k^2}{1+k^2}\, s\right) K_0(s)\, ds$$

Using Eqs. (1) and (24) this may be written

$$\Psi(k, u) = \frac{\tan^{-1}(k)}{\pi^{1/2}} + \frac{1}{\pi^{1/2}} \frac{k}{1+k^2} K_{e_0}\left[\frac{1-k^2}{1+k^2}, \frac{1}{2}u^2(1+k^2)\right]$$

We may then write Eq. (23)

$$D(y, \epsilon) = \frac{\epsilon}{2\pi^{3/2}\eta\sigma} \exp\left(\frac{-y^2}{8\epsilon^2\eta^2\sigma^2}\right) K_0\left(\frac{y^2}{8\epsilon^2\eta^2\sigma^2}\right)$$

$$+ \frac{\sqrt{1-\epsilon^2}}{\pi^{3/2}\eta\sigma} \exp\left(\frac{-y^2}{4\eta^2\sigma^2}\right) [\cos^{-1}\epsilon + \epsilon\sqrt{1-\epsilon^2}\, K_{e_0}(2\epsilon^2 - 1, y^2/8\epsilon^2\eta^2\sigma^2)]$$

where $K_{e_0}(a, z)$ is given by Eq. (18).

SOME INTEGRALS RELATED TO $K_{e_0}(a, z)$

The following integrals can easily be obtained from Eq. (17):

$$\int_0^1 \sin(\alpha t) K_0(\beta t)\, dt = \frac{\sinh^{-1}(\alpha/\beta)}{\sqrt{\alpha^2+\beta^2}} - \frac{\alpha}{\beta} K_1(\beta)\, M_0(-\alpha^2/4, -\alpha^2/\beta^2)$$

$$+ \frac{\alpha^3}{3\beta^2} K_0(\beta)\, M_1(-\alpha^2/4, -\alpha^2/\beta^2) \qquad |\alpha/\beta| \leqslant 1,\ \alpha \neq \pm i\beta$$

$$\int_0^1 \cos(\alpha t)\, K_0(\beta t)\, dt = \beta K_1(\beta)\, L_0(-\alpha^2/4, \beta^2/4) + K_0(\beta) L_1(-\alpha^2/4, \beta^2/4)$$

Further, using the result [2] for $0 < |\alpha| < 1, 0 < x$

$$K_{e_0}(\alpha, x) = \operatorname{sgn}\alpha \left\{\exp(\alpha x)\left[\int_0^\infty \frac{\cos(\alpha x t)\, dt}{(1+t^2)\sqrt{1+\alpha^2 t^2}} + \int_0^\infty \frac{t \sin(\alpha x t)\, dt}{(1+t^2)\sqrt{1+\alpha^2 t^2}}\right] - \frac{\cos^{-1}(|\alpha|)}{\sqrt{1-\alpha^2}}\right\}$$

we find for $0 < \alpha < \beta$

$$\int_0^\infty \frac{\cos(\alpha x)\, dx}{(1+x^2)\sqrt{\beta^2+\alpha^2 x^2}} = \cosh\alpha \left\{\frac{\pi/2}{\sqrt{\beta^2-\alpha^2}} - \frac{\alpha}{\beta} K_1(\beta)\, M_0(\alpha^2/4, \alpha^2/\beta^2) - \frac{\alpha^3}{3\beta^2} K_0(\beta)\, M_1(\alpha^2/4, \alpha^2/\beta^2)\right\}$$

$$- \sinh\alpha \left\{\beta K_1(\beta) L_0(\alpha^2/4, \beta^2/4) + K_0(\beta) L_1(\alpha^2/4, \beta^2/4)\right\}$$

$$\int_0^\infty \frac{x \sin(\alpha x)\,dx}{(1+x^2)\sqrt{\beta^2+\alpha^2x^2}} = \cosh\alpha \left\{\beta K_1(\beta) L_0(\alpha^2/4, \beta^2/4) + K_0(\beta) L_1(\alpha^2/4, \beta^2/4)\right\}$$

$$-\sinh\alpha \left\{\frac{\pi/2}{\sqrt{\beta^2-\alpha^2}} - \frac{\alpha}{\beta} K_1(\beta) M_0(\alpha^2/4, \alpha^2/\beta^2) - \frac{\alpha^3}{3\beta^2} K_0(\beta) M_1(\alpha^2/4, \alpha^2/\beta^2)\right\}$$

In addition [9, Eq. (3.367), p. 316] we have

$$\int_0^\infty \frac{e^{-pt} \sin\theta\,dt}{(1+t+\cos\theta)\sqrt{t^2+2t}} = \exp\left\{2p\cos^2\frac{\theta}{2}\right\} [\theta - \sin\theta\, K_{e_0}(-\cos\theta, p)] \qquad \mathrm{Re}\, p > 0$$

CONCLUSIONS

The Kampé de Fériet functions have been used to put in closed form the incomplete Lipschitz-Hankel integral $K_{e_0}(a, z)$ and several related integrals that are not readily available and are of interest in mathematical physics and applications. Some of the properties of the Kampé de Fériet functions associated with $K_{e_0}(a, z)$ are derived. These properties are useful in deriving additional results quickly. As an example we have given an elementary derivation of a closed form for King's integral based on generating function techniques.

In addition, the utility of a closed form for $K_{e_0}(a, z)$ is indicated by deriving a certain density function that is associated with the scattered coherent return from the sea surface.

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